HL Paper 1 Mock B 2021 – WORKED SOLUTIONS v2

Section A

1. (a) $h(x) = (f \circ g)(x) = f(g(x))$

$f(g(x)) = f(2x-3)$

$$= \frac{1}{2(2x-3)+1}$$

$$= \frac{1}{4x-6+1}$$

$$f(g(x)) = \frac{1}{4x-5}$$

thus, $h(x) = \dfrac{1}{4x-5}$

(b) $y = \dfrac{1}{4x-5}$; solve for x

$$\frac{1}{y} = 4x-5 \;\Rightarrow\; 4x = \frac{1}{y}+5 \;\Rightarrow\; x = \frac{1}{4y}+\frac{5}{4}$$

thus, $h^{-1}(x) = \dfrac{1}{4x}+\dfrac{5}{4}$ or, equivalently, $h^{-1}(x) = \dfrac{5x+1}{4x}$

2. (a) $g'(x) = (x-4)^3$; find $g''(x)$ using the chain rule

let $u = x-4$ and let $y = u^3$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$g''(x) = \frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = 3u^2\cdot 1$$

undo substitution:

$$g''(x) = 3(x-4)^2$$

(b) $g''(4) = 3(4-4)^2 = 0$

(c) $g''(x) \geq 0$ for all values of x, meaning that the graph of $g(x)$ is always concave up. An inflexion point requires a change in sign of the value of $g''(x)$, i.e. it requires the graph of $g(x)$ to go from concave up to concave down, or vice versa. As $g(x)$ is always concave up, neither point A nor any other point on $g(x)$ is an inflexion point.

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3. (a) $\log_3 20 = \log_3(4\cdot 5) = \log_3(2^2) + \log_3 5 = 2\log_3 2 + \log_3 5 = 2x + y$

thus, $\log_3 20 = 2x + y$

(b) $\log_3\left(7\tfrac{13}{16}\right) = \log_3\left(\dfrac{7\cdot 16 + 13}{16}\right) = \log_3\left(\dfrac{112+13}{16}\right) = \log_3\left(\dfrac{125}{16}\right)$

$\log_3\left(\dfrac{125}{16}\right) = \log_3 125 - \log_3 16 = \log_3(5^3) - \log_3(2^4) = 3\log_3 5 - 4\log_3 2 = 3y - 4x$

thus, $\log_3\left(7\tfrac{13}{16}\right) = 3y - 4x$

(c) $\log_5 8 = \dfrac{\log_3 8}{\log_3 5} = \dfrac{\log_3(2^3)}{\log_3 5} = \dfrac{3\log_3 2}{\log_3 5} = \dfrac{3x}{y}$

thus, $\log_5 8 = \dfrac{3x}{y}$

4. (a) $1 + \ln x + (\ln x)^2 + \cdots$ is a geometric sequence with $u_1 = 1$ and $r = \ln x$

a geometric sequence converges when $|r| < 1$

$|\ln x| < 1 \;\Rightarrow\; -1 < \ln x < 1$

$\ln x < 1 \;\Rightarrow\; x < \mathrm{e}$

$\ln x > -1 \;\Rightarrow\; x > \mathrm{e}^{-1}$

thus, the sequence converges when $\dfrac{1}{\mathrm{e}} < x < \mathrm{e}$

(b) the sum of an infinite geometric sequence is $S_\infty = \dfrac{u_1}{1-r}$

for this sequence, $u_1 = 1$ and $r = \ln x$

$S_\infty = \dfrac{1}{1-\ln x}$; we want to find the value of x such that $S_\infty = 2$

$\dfrac{1}{1-\ln x} = 2 \;\Rightarrow\; 1 - \ln x = \dfrac{1}{2} \;\Rightarrow\; \ln x = \dfrac{1}{2} \;\Rightarrow\; x = \mathrm{e}^{\frac{1}{2}} = \sqrt{\mathrm{e}}$

thus, the sequence converges to 2 when $x = \sqrt{\mathrm{e}}$

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5. (a) At point $P(p,-1): 7(-1)^3 + p(-1)^2 - p^2(-1) + 1 = 0 \quad \Rightarrow \quad p^2 + p - 6 = 0$

$(p-2)(p+3) = 0 \quad \Rightarrow \quad p = 2$ or $p = -3$

Hence, coordinates of P are either $(2,-1)$ or $(-3,-1)$

find $\frac{dy}{dx}$ by implicit differentiation: $\frac{d}{dx}\left(7y^3 + xy^2 - x^2y + 1\right) = \frac{d}{dx}(0)$

$$21y^2 \cdot \frac{dy}{dx} + \left(y^2 + 2xy \cdot \frac{dy}{dx}\right) - \left(2xy + x^2 \cdot \frac{dy}{dx}\right) = 0$$

$21y^2 \cdot \frac{dy}{dx} + y^2 + 2xy \cdot \frac{dy}{dx} - 2xy - x^2 \cdot \frac{dy}{dx} = 0$; solve for $\frac{dy}{dx}$

$$\left(21y^2 + 2xy - x^2\right)\frac{dy}{dx} = 2xy - y^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2xy - y^2}{21y^2 + 2xy - x^2}$$

At $(2,-1)$: $\frac{dy}{dx} = \frac{2(2)(-1) - (-1)^2}{21(-1)^2 + 2(2)(-1) - (2)^2} == \frac{-4-1}{21-4-4} = -\frac{5}{13} \neq \frac{5}{18}$ thus, $p \neq 2$

At $(-3,-1)$: $\frac{dy}{dx} = \frac{2(-3)(-1) - (-1)^2}{21(-1)^2 + 2(-3)(-1) - (-3)^2} == \frac{6-1}{21+6-9} = \frac{5}{18}$ therefore, $p = -3$

6. $8\sin x \cos x = \sqrt{12}$; simplify LHS using trigonometric identities

$\text{LHS} = 8\sin x \cos x = 4(2\sin x \cos x) = 4\sin 2x$

simplify RHS:

$\text{RHS} = \sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$

$$4\sin 2x = 2\sqrt{3} \quad \Rightarrow \quad \sin 2x = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

we are told to solve the equation for $0 \leq x \leq \frac{\pi}{2}$, i.e. $0 \leq 2x \leq \pi$

therefore, $2x = \frac{\pi}{3}, \frac{2\pi}{3}$

hence, $x = \frac{\pi}{6}, \frac{\pi}{3}$

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7. α and β are factors of the equation $2x^2 - x + 4 = 0$, therefore:

$$2(x-\alpha)(x-\beta)=0 \;\Rightarrow\; 2x^2 - 2(\alpha+\beta)x + 2\alpha\beta = 0$$

equating $2x^2 - x + 4$ to $2x^2 - 2(\alpha+\beta)x + 2\alpha\beta$, we find that:

$$-2(\alpha+\beta) = -1 \;\Rightarrow\; \alpha+\beta = \frac{1}{2} \text{ and } 2\alpha\beta = 4 \;\Rightarrow\; \alpha\beta = 2$$

the factors of the new equation are $\alpha+2$ and $\beta+2$, so write the new equation in the form

$$a\left(x-(\alpha+2)\right)\left(x-(\beta+2)\right)=0,\; a\in\mathbb{R}$$

expanding out:

$$ax^2 - a(\alpha+\beta+4)x + a(\alpha+2)(\beta+2) = 0$$

we know that $\alpha+\beta = \frac{1}{2}$, therefore $\alpha+\beta+4 = 4+\frac{1}{2} = \frac{9}{2}$

similarly, $(\alpha+2)(\beta+2) = \alpha\beta + 2(\alpha+\beta) + 4 = 2 + 2\left(\frac{1}{2}\right) + 4 = 7$

so, the new equation becomes:

$$ax^2 - \frac{9}{2}ax + 7a = 0$$

setting the value of a as 2, we have a quadratic equation with integer coefficients

thus, a quadratic equation with integer coefficients whose roots are $\alpha+2$ and $\beta+2$ is

$$2x^2 - 9x + 14 = 0$$

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8. Show statement is true for $n=1$:

$4^1+2=6=2(3)$; thus, statement is true for $n=1$

Assume that the statement is true for a specific value of n, call it k. That is, assume

$4^k+2=3M$ where M is a positive integer

Show that it must follow that the statement is true for $n=k+1$. That is, show that $4^{k+1}+2$ must be a multiple of 3.

$4^{k+1}+2=4\cdot 4^k+2$

From the assumption, it follows that $4^k=3M-2$. Substituting this, gives

$$4^{k+1}+2=4(3M-2)+2$$
$$=12M-6$$
$$=3(4M-2)$$

Since M is a positive integer then $4M-2$ must be a positive integer. And, since it was shown that $4^{k+1}+2=3(4M-2)$ then $4^{k+1}+2$ is a multiple of 3.

Hence, by the principle of mathematical induction, the expression 4^n+2 must be a multiple of 3 for all positive integer values of n.

9. Area of sector: $A=\frac{1}{2}r^2\theta$

differentiate implicitly with respect to t:

$$\frac{dA}{dt}=\frac{d}{dt}\left(\frac{1}{2}r^2\theta\right)$$
$$=\frac{1}{2}\cdot 2r\cdot\frac{dr}{dt}\cdot\theta+\frac{1}{2}\cdot r^2\cdot\frac{d\theta}{dt}$$
$$\frac{dA}{dt}=r\theta\frac{dr}{dt}+\frac{1}{2}r^2\frac{d\theta}{dt}$$

given that the area of the sector is increasing at a rate of 2π cm^2 per second, i.e. $\frac{dA}{dt}=2\pi$, and

the radius is increasing at a rate of 2 cm per second, i.e. $\frac{dr}{dt}=2$

substituting these values, along with $r=3$ and $\theta=\frac{\pi}{4}$, into the above equation gives

$$2\pi=3\cdot\frac{\pi}{4}\cdot 2+\frac{1}{2}\cdot 3^2\cdot\frac{d\theta}{dt} \quad\Rightarrow\quad 2\pi-\frac{3\pi}{2}=\frac{9}{2}\frac{d\theta}{dt}$$

solving for $\frac{d\theta}{dt}$: $\frac{d\theta}{dt}=\frac{\pi}{2}\cdot\frac{2}{9}=\frac{\pi}{9}$

thus, when $r=3$ and $\theta=\frac{\pi}{4}$, $\frac{d\theta}{dt}=\frac{\pi}{9}$ radians per second

[also accept $\frac{d\theta}{dt}=20$ degrees per second]

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Section B

10. (a) $g(x)=\dfrac{3x}{x^2+7}$; find $g'(x)$ using quotient rule

let $u=3x \;\Rightarrow\; \dfrac{\mathrm{d}u}{\mathrm{d}x}=3$

let $v=x^2+7 \;\Rightarrow\; \dfrac{\mathrm{d}v}{\mathrm{d}x}=2x$

$$g'(x)=\frac{3\cdot\left(x^2+7\right)-3x\cdot 2x}{\left(x^2+7\right)^2}$$

$$=\frac{3x^2+21-6x^2}{\left(x^2+7\right)^2}$$

$$g'(x)=\frac{21-3x^2}{\left(x^2+7\right)^2}$$ **Q.E.D.**

(b) $\displaystyle\int\frac{3x}{x^2+7}\,\mathrm{d}x$; using integration by substitution

let $u=x^2+7 \;\Rightarrow\; \dfrac{\mathrm{d}u}{\mathrm{d}x}=2x$

$$\int\frac{3x}{x^2+7}\,\mathrm{d}x=\frac{3}{2}\int\frac{1}{u}\,\mathrm{d}u=\frac{3}{2}\ln u+C$$

Substituting x^2+7 back in for *u* gives: $\displaystyle\int\frac{3x}{x^2+7}\,\mathrm{d}x=\frac{3}{2}\ln\left(x^2+7\right)+C$

(c) area $=\displaystyle\int_{\sqrt{7}}^{a}g(x)\,\mathrm{d}x$

$$=\int_{\sqrt{7}}^{a}\frac{3x}{x^2+7}\,\mathrm{d}x$$

from (b), $\displaystyle\int\frac{3x}{x^2+7}\,\mathrm{d}x=\frac{3}{2}\ln\left(x^2+7\right)+C$

thus, area $=\left[\dfrac{3}{2}\ln\left(x^2+7\right)\right]_{\sqrt{7}}^{a}$

$$=\frac{3}{2}\ln\left(a^2+7\right)-\frac{3}{2}\ln\left(\left(\sqrt{7}\right)^2+7\right)$$

$$=\frac{3}{2}\ln\left(a^2+7\right)-\frac{3}{2}\ln 14$$

$$=\frac{3}{2}\ln\left(\frac{a^2+7}{14}\right)$$

[solution for Q.10 is continued on the next page]

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[solution for Q.10 continued]

given that the area of the region is $\ln 8$

hence, $\frac{3}{2}\ln\left(\frac{a^2+7}{14}\right)=\ln 8$

$\ln\left(\frac{a^2+7}{14}\right)=\frac{2}{3}\ln 8=\ln\left(8^{\frac{2}{3}}\right)==\ln\left(\left(2^3\right)^{\frac{2}{3}}\right)=\ln\left(2^2\right)$

$\ln\left(\frac{a^2+7}{14}\right)=\ln 4$

$\frac{a^2+7}{14}=4 \Rightarrow a^2+7=56$

$a^2=56-7=49 \Rightarrow a=\pm 7$

$a>\sqrt{7}$, therefore $a=7$

11. (a) Let $z=a+b\text{i},\ a,\ b\in\mathbb{R}$, then $z-3\text{i}=a+(b-3)\text{i}$

the imaginary part of z is b

$|z|=\sqrt{a^2+b^2} \Rightarrow |z-3\text{i}|=\sqrt{a^2+(b-3)^2}$

$\sqrt{a^2+b^2}=\sqrt{a^2+(b-3)^2} \Rightarrow a^2+b^2=a^2+(b-3)^2$

$b^2=b^2-6b+9 \Rightarrow 6b=9 \Rightarrow b=\frac{3}{2}$

thus, the imaginary part of z is $\frac{3}{2}$ **Q.E.D.**

(b) (i) $|z|=\sqrt{a^2+b^2}=3 \Rightarrow a^2+b^2=9$; substitute $b=\frac{3}{2}$

$a^2+\frac{9}{4}=9 \Rightarrow a^2=\frac{27}{4} \Rightarrow a=\pm\frac{3\sqrt{3}}{2}$

let $z_1=\frac{3\sqrt{3}}{2}+\frac{3}{2}\text{i}$ and let $z_2=-\frac{3\sqrt{3}}{2}+\frac{3}{2}\text{i}$

[solution for Q.11 is continued on the next page]

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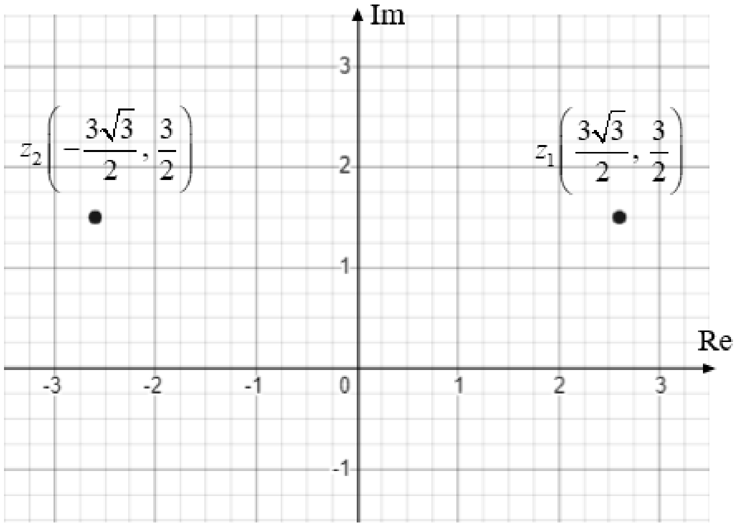
[solution for Q.11 continued]



(ii) $\arg z_1$ is the angle between the real axis and the line that passes through $(0, 0)$ and z_1 in the graph in (i)

$$\tan(\arg z_1) = \frac{\frac{3}{2}}{\frac{3\sqrt{3}}{2}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\arg z_1 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

(iii) $\arg z_2 = \pi - \arg z_1 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

(c) $\arg\left(\frac{z_1{}^k z_2}{2\mathrm{i}}\right) = \frac{k\pi}{6} + \frac{5\pi}{6}$

$$\arg\left(z_1{}^k\right) + \arg(z_2) - \arg(2\mathrm{i}) = \pi$$

$$\frac{\pi}{6}k + \frac{5}{6}\pi - \frac{\pi}{2} = \pi$$

Thus, $k = 4$

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[solution for Q.11 continued]

(d) Applying $\ln a + \ln b = \ln(ab)$:

$$\ln(x^2) + \ln\left(\frac{x^2}{y}\right) + \ln\left(\frac{x^2}{y^2}\right) + \ln\left(\frac{x^2}{y^3}\right) + \cdots = \ln\left(\frac{x^2}{y^0} \cdot \frac{x^2}{y^1} \cdot \frac{x^2}{y^2} \cdot \frac{x^2}{y^3} \cdot \ldots\right)$$

In the sum of the first 20 terms, there are 20 x^2 terms in the logarithm, so:

$$\ln\left(\frac{x^2}{y^0} \cdot \frac{x^2}{y^1} \cdot \frac{x^2}{y^2} \cdot \frac{x^2}{y^3} \cdot \ldots\right) = \ln\left(\frac{(x^2)^{20}}{y^0 \cdot y^1 \cdot y^2 \cdot y^3 \cdot \ldots}\right) = \ln\left(\frac{x^{40}}{y^0 \cdot y^1 \cdot y^2 \cdot y^3 \cdot \ldots}\right)$$

considering only the denominator in the logarithm:

$$y^0 \cdot y^1 \cdot y^2 \cdot y^3 \cdot \ldots$$

As we are summing the first 20 terms, there will be 20 y terms, each with a power that is one greater than that of the last

Since the first y term has a power of 0, the 20th y term will have a power of 19

$$y^0 \cdot y^1 \cdot y^2 \cdot y^3 \cdot \ldots \cdot y^{18} \cdot y^{19}$$

Simplifying:

$$y^0 \cdot y^1 \cdot y^2 \cdot y^3 \cdot \ldots \cdot y^{18} \cdot y^{19} = y^{(0+1+2+3+\ldots+18+19)}$$

This power of y is equal to the sum of an arithmetic sequence with $u_1 = 0$ and $u_{20} = 19$

Using the formula for the sum of an arithmetic sequence:

$$S_{20} = \frac{20}{2}(0+19) = 190$$

Therefore, $y^{(0+1+2+3+\ldots+18+19)} = y^{190}$

Substituting gives:

$$\ln\left(\frac{x^{40}}{y^0 \cdot y^1 \cdot y^2 \cdot y^3 \cdot \ldots}\right) = \ln\left(\frac{x^{40}}{y^{190}}\right)$$

Therefore, the sum of the first 20 terms of the sequence is $\ln\left(\frac{x^{40}}{y^{190}}\right)$

that is, $\ln(x^2) + \ln\left(\frac{x^2}{y}\right) + \ln\left(\frac{x^2}{y^2}\right) + \ln\left(\frac{x^2}{y^3}\right) + \cdots = \ln\left(\frac{x^{40}}{y^{190}}\right)$

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12. (a) $\cos 2\theta = 2\cos^2\theta - 1 \quad \Rightarrow \quad 2\cos^2\theta = 1 + \cos 2\theta \quad \Rightarrow \quad \cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ **Q.E.D.**

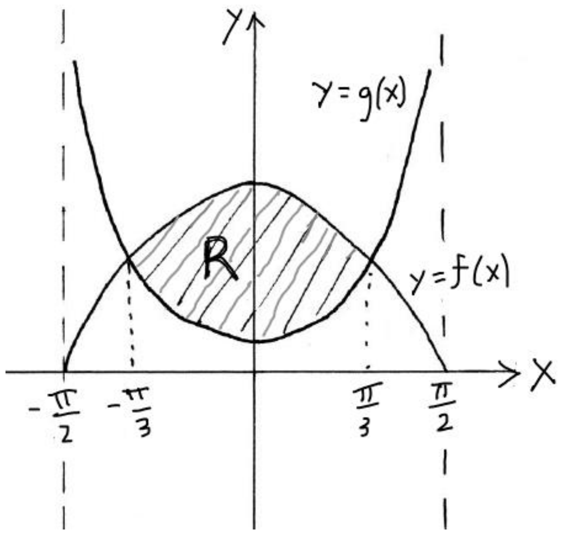
(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int \cos 2x\,dx + \int \frac{1}{2}\,dx$$
$$= \frac{1}{4}\sin 2x + \frac{1}{2}x + C$$

(c) The graphs of f and g intersect when $f(x) = g(x)$ in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$4\cos x = \sec x \quad \Rightarrow \quad 4\cos x = \dfrac{1}{\cos x} \quad \Rightarrow \quad \cos^2 x = \dfrac{1}{4}$

$\cos x = \pm\dfrac{1}{2} \quad \Rightarrow \quad x = \pm\dfrac{\pi}{3}$ thus, f and g intersect at $x = -\dfrac{\pi}{3}$ and at $x = \dfrac{\pi}{3}$

(d)



(e) (i) Substituting into $V = \pi\int_a^b y^2\,dx$ gives:

Volume $= \pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(16\cos^2 x - \sec^2 x\right)dx$

(ii) $$\pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(16\cos^2 x - \sec^2 x\right)dx = \pi\left[16\left(\frac{1}{4}\sin 2x + \frac{1}{2}x\right) - \tan x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$
$$= \pi\left[\left(4\sin\frac{2\pi}{3} + \frac{8\pi}{3} - \tan\frac{\pi}{3}\right) - \left(4\sin\left(-\frac{2\pi}{3}\right) - \frac{8\pi}{3} - \tan\left(-\frac{\pi}{3}\right)\right)\right]$$
$$= \pi\left[\left(2\sqrt{3} + \frac{8\pi}{3} - \sqrt{3}\right) - \left(-2\sqrt{3} - \frac{8\pi}{3} + \sqrt{3}\right)\right] = \pi\left(4\sqrt{3} + \frac{16\pi}{3} - 2\sqrt{3}\right)$$
$$= \pi\left(2\sqrt{3} + \frac{16\pi}{3}\right)$$
$$= 2\pi\left(\sqrt{3} + \frac{8\pi}{3}\right) \text{ units}^3$$